# Coordinate–Free Quantization of Second–Class Constraints


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**Abstract**

The conversion of second-class constraints into first-class constraints is used to extend the coordinate-free path integral quantization, achieved by a flat-space Brownian motion regularization of the coherent-state path integral measure, to systems with second-class constraints


## 1 Second-class constraints

On performing the Legendre transformation for generalized velocities of a Lagrangian of a dynamical system, one very often gets relations between canonical coordinates and momenta that do not involve time derivatives. They are therefore not equations of motion and are called primary constraints [1]. The primary constraints should be satisfied as time proceeds, which leads to further conditions on dynamical variables known as secondary constraints [1].

Let $\varphi_a = \varphi_a(\theta) = 0$ be all independent constraints (primary and secondary) in the system; here $\theta^i,\ i = 1, 2, ..., 2N$, denote canonical variables that span a Euclidean phase space of the system. The canonical symplectic structure is assumed on the phase space $\{\theta^i, \theta^j\} = \overset{\circ}{\omega}{}^{ij}$; one can, for instance, set $q^n = \theta^{2n-1}$ and $p_n = \theta^{2n},\ n = 1, 2, ..., N$ for canonical coordinates and their momenta, then $\{p_n, q^m\} = \delta_n^m$ and other components of the canonical symplectic structure are zero. Let $H(\theta)$ be the canonical Hamiltonian of the system. Since $\varphi_a$ is a complete set of constraints,

$$\dot{\varphi}_a = \{\varphi_a, H\} = C_a^b(\theta)\varphi_b \approx 0 \ , \tag{1.1}$$

where the symbol $\approx$ implies the weak equality [1] that is valid on the constraint surface $\varphi_a = 0$.

Systems with constraints admit a more general dynamical description where the Hamiltonian can be replaced by a generalized one $H_T = H + \lambda^a(\theta, t)\varphi_a(\theta)$ with $\lambda^a$ being arbitrary functions of $\theta^i$ and time:

$$\dot{\theta}^i = \{\theta^i, H_T\} \approx \{\theta^i, H\} + \lambda^a\{\theta^i, \varphi_a\} \ . \tag{1.2}$$

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The condition (1.1) with $H$ replaced by $H_T$ enforces some restrictions on $\lambda^a$. Indeed

$$\dot{\varphi}_a \approx \{\varphi_a, \varphi_b\}\lambda^b = 0 \ , \tag{1.3}$$

that is, the number of independent functions $\lambda^a$ is determined by the rank of the matrix $\{\varphi_a, \varphi_b\} = \Delta_{ab}$ on the surface of constraints. If the rank is zero, i.e. $\{\varphi_a, \varphi_b\} = C^c_{ab}\varphi_c \approx 0$ then all the $\lambda^a$ are arbitrary, and the solutions $\theta^i(t)$ to the equations of motion (1.2) would contain arbitrary functions $\lambda^a$. Such constraints are called first-class constraints [1] and transformations of $\theta^i(t)$ generated by $\lambda^a \to \lambda^a + \delta\lambda^a$ are known as gauge transformations. The latter implies in particular that the dynamical system has more non-physical canonical variables in addition to those fixed by the constraints $\varphi_a = 0$. They can be removed by specifying $\lambda^a$ (by gauge fixing).

If the rank of $\Delta_{ab}$ on the surface $\varphi_a = 0$ is equal to the number of independent constraints then $\det \Delta_{ab} \not\approx 0$ and all the $\lambda_a$ must be zero. The system admits no gauge arbitrariness. Such constraints are with an arbitrarily large diffusion constant called second-class constraints [1]. Of course there could well be a "mixed" case when the system possesses second and first class constraints (the matrix $\Delta_{ab}$ is degenerate, but $\Delta_{ab} \not\approx 0$). In what follows only second-class constraints are considered.

The number of second-class constraints must be even because the determinant of an antisymmetric matrix of odd order is always zero. So we set $a = 1, 2, \ldots, 2M$, that is the system has only $2(N-M)$ physical canonical variables that describe dynamics on the physical phase space determined by $\varphi_a(\theta) = 0$.

## 2 Local parametrizations of the physical phase space

One can introduce a local parametrization of the constraint surface $\theta^i = \theta^i(\vartheta)$ where $\vartheta^\alpha, \alpha = 1, 2, \ldots, 2(N-M)$, are chosen so that $\varphi_a(\theta(\vartheta))$ identically vanish for all values of $\vartheta^\alpha$. The local coordinates $\vartheta^\alpha$ span the physical phase space. They may also serve as local symplectic variables on the physical phase space, provided there is an induced symplectic structure on it. To obtain an induced symplectic structure for an ordinary change of variables on phase space, one would have to invert the relations $\theta^i = \theta^i(\vartheta)$ and calculate the Poisson bracket of $\vartheta^\alpha$. This procedure is incorrect for systems with constraints. The inverse relations $\vartheta^\alpha = \vartheta^\alpha(\theta)$ are determined modulo the constraints $\varphi_a = 0$. Since $\{\theta^i, \varphi_a\} \neq 0$, the Poisson bracket $\{\vartheta^\alpha, \vartheta^\beta\}$ is ambiguous and does not induce the right symplectic structure. Recall that due to the same reason ($\{\theta^i, \varphi^a\} \neq 0$), the constraints should not be solved before calculating the Poisson bracket in equations of motion (1.2).

The problem is resolved by means of the Dirac bracket that reads as [1]

$$\{A, B\}_D = \{A, B\} - \{A, \varphi_a\}\Delta^{ab}\{\varphi_b, B\} \tag{2.1}$$

for any $A$ and $B$, and $\Delta_{ab}\Delta^{bc} = \delta^c_a$. The Dirac bracket possesses three important properties [1]. First, it satisfies the Jacobi identity, the Leibnitz rule and is antisymmetric, therefore it determines a symplectic structure

$$\{\theta^i, \theta^j\}_D = \overset{\circ}{\omega}{}^{ij} - \{\theta^i, \varphi_a\}\Delta^{ab}\{\varphi_b, \theta^j\} \equiv \omega^{ij}_D(\theta) \ . \tag{2.2}$$

Second, it vanishes for any $A(\theta)$ and any of the constraints $\varphi_a$,

$$\{A, \varphi_a\}_D = 0 \ . \tag{2.3}$$

As a consequence of (2.3) we deduce the third property

$$\dot{A} = \{\dot{A}, H\}_D \approx \{A, H\} \ , \tag{2.4}$$

i.e. the equations of motion are not affected by the replacement of the Poisson bracket by the Dirac one.

The advantage of using the Dirac bracket is that one can solve the constraints at any stage of calculation, before or after calculating the brackets. The latter is guaranteed by (2.3). In particular, given a set of local parameters $\vartheta^\alpha = \vartheta^\alpha(\theta)$ spanning the surface $\varphi^a = 0$, the induced symplectic structure is unique

$$\{\vartheta^\alpha, \vartheta^\beta\}_D = \omega^{\alpha\beta}(\vartheta) \ . \tag{2.5}$$

We remark that the Dirac symplectic structure (2.2) is degenerate. Its non-degenerate part is given by (2.5) in local coordinates $\vartheta^\alpha$. There is an infinite number of choices of local symplectic coordinates on the physical phase space. One can find such a parametrization for which the induced symplectic structure (2.5) has the canonical form $\omega^{\alpha\beta} = \stackrel{\circ}{\omega}{}^{\alpha\beta}$. The latter follows from the Darboux theorem, and the corresponding $\vartheta^\alpha$ are Darboux coordinates for the symplectic structure (2.5). But even the Darboux coordinates are not unique, since they are determined up to a canonical transformation.

Thus, classical dynamics of second-class constrained systems exhibits a physical-phase-space reparametrization invariance.

# 3 Examples of second-class constraints

Consider a Lagrangian of the form

$$L = \frac{1}{2}\dot{\mathbf{x}}^2 + yF(\mathbf{x}) \ , \tag{3.1}$$

where $\mathbf{x}$ is a radius vector in $\mathbb{R}^N$. It describes a motion of a point-like particle of unit mass on an $N-1$- dimensional surface determined by the equation $F(\mathbf{x}) = 0$, as follows from the Euler-Lagrange equation for the Lagrange multiplier $y$. The canonical momentum for $y$ vanishes yielding the primary constraint

$$p_y = \frac{\partial L}{\partial \dot{y}} = 0 \ . \tag{3.2}$$

Doing the Legendre transformation for $\dot{\mathbf{x}}$, we arrive at the Hamiltonian

$$H = \frac{1}{2}\mathbf{p}^2 - yF(\mathbf{x}) + \lambda p_y \ , \tag{3.3}$$

where $\mathbf{p}$ is the canonical momentum for $\mathbf{x}$, and $\lambda$ is an arbitrary function of canonical variables and time. Its occurrence in (3.3) is due to (3.2) (since $p_y$ vanishes identically, the canonical Hamiltonian is determined up to any function that vanishes as $p_y = 0$).

To find secondary constraints, one should calculate the Poisson bracket of the primary constraint (3.2) with the Hamiltonian (3.3)

$$\dot{p}_y = \{p_y, H\} \approx F(\mathbf{x}) = 0 \ , \tag{3.4}$$

that is, the motion is indeed constrained to the surface $F = 0$. We should continue checking the dynamical self-consistency for the constraint (3.4)

$$\dot{F}(x) = \{F, H\} \approx (\mathbf{p}, \boldsymbol{\partial} F) = 0 \ . \tag{3.5}$$

Equation (3.5) determines a new constraint. Let us denote the constraints (3.2), (3.4) and (3.5) as $\varphi_{1,2,3}$, respectively. Then there must be

$$\dot{\varphi}_3 = \{\varphi_3, H\} \approx p_i p_j \partial_i \partial_j F + y(\boldsymbol{\partial} F)^2 = 0 \tag{3.6}$$

The new constraint (3.6) is denoted by $\varphi_4$. The theory does not have more constraints because the condition $\dot{\varphi}_4 = \{\varphi_4, H\} \approx 0$ yields an equation for an arbitrary function $\lambda$, rather than for canonical variables. It is not hard to be convinced that all four independent constraints $\varphi_a$ are of the second class, i.e. $\det \Delta_{ab} = \det\{\varphi_a, \varphi_b\} \not\approx 0$.

A geometrical meaning of (3.4) and (3.5) is transparent. Equation (3.4) implies that the particle moves along the surface $F = 0$. Equation (3.5) means that the particle momentum remains tangent to the surface $F = 0$ during the motion. Note that the vector $\boldsymbol{\partial} F$ is locally transverse to the surface $F = 0$. The constraints (3.2) and (3.6) are artifacts of constructing the Hamiltonian formalism from the Lagrangian (3.1) where $y$ is not a dynamical variable, rather it is a Lagrange multiplier used to enforce the constraint (3.4) in the Lagrangian formalism. Since in the Legendre transformation the variable $y$ is treated as an independent dynamical variable, the associated Hamiltonian formalism exhibit two extra constraints (3.2) and (3.6) to suppress dynamics of $y$ and $p_y$.

In fact, we may start right from the Hamiltonian formalism (3.3), setting in it $\lambda \equiv 0$ and $y$ to be an arbitrary function of $\mathbf{p}$ and $\mathbf{x}$. Then the constraint (3.4) should be regarded as the "primary" constraint. In this simplified approach equation (3.6) is not a constraint, but the equation for an arbitrary function $y$ (the Lagrange multiplier). The Dirac bracket formalism leads to the same answer for the symplectic structure on the physical phase space, so we prefer the simplified Hamiltonian formalism. In general, given a set of the second class constraints $\varphi_a$ to be imposed on the motion generated by the Hamiltonian $H_s$ of a system under consideration, one can consider a generalized Hamiltonian formalism

$$H = H_s(\theta) + \lambda^a(\theta, t)\varphi_a(\theta) \ . \tag{3.7}$$

The consistency conditions $\dot{\varphi}_a = \{\varphi_a, H\} \approx 0$ yield equations for arbitrary functions $\lambda_a$ whose solutions are

$$\lambda^a = \Delta^{ab}\{\varphi_b, H_s\} \ . \tag{3.8}$$

For instance, to describe the motion along a surface $F = 0$ in the Hamiltonian formalism, one can take $H_s = \mathbf{p}^2/2$ and two constraints $\varphi_1 = F$ and $\varphi_2 = (\mathbf{p}, \boldsymbol{\partial} F)$ and consider the generalized Hamiltonian dynamics (3.7).

A symplectic structure on the physical phase space determined by the constraints $\varphi_{1,2} = 0$ is induced by the Dirac bracket

$$\{x_i, x_j\}_D = 0 \; ; \tag{3.9}$$

$$\{x_i, p_j\}_D = \delta_{ij} - n_i n_j \; ; \tag{3.10}$$

$$\{p_i, p_j\}_D = p_k \left(n_j \partial_k n_i - n_i \partial_k n_j\right) \; , \tag{3.11}$$

where $n_i = n_i(\mathbf{x}) = \partial_i F/|\partial F|$ is a unit vector that coincides with the normal to the surface when $\mathbf{x}$ is on the surface. The symplectic structure is by construction degenerate. To construct the induced (non-degenerate) symplectic structure on the physical phase space, one should introduce a local parametrization of the constraint surface $\varphi_{1,2} = 0$ and calculate the Dirac bracket for local coordinates spanning the constraint surface. For instance, given a local parametrization of the surface $F(\mathbf{x}) = 0$ in the form $\mathbf{x} = \mathbf{x}^F(u),\; u \in \mathrm{I\!R}^{N-1}$, the physical momenta are $p_i = e_i^a(u) p_{u_a}$ where the vectors $\mathbf{e}^a(u)$ form a basis in the tangent space of the surface. In particular, one can take $e_i^a(u) = \partial x_i^F/\partial u_a$. From the identity $F(\mathbf{x}^F) \equiv 0$ follows the orthogonality relation $(\mathbf{n}(\mathbf{x}^F), \mathbf{e}^a) = 0$. The variables $p_{u_a}, u_a$ serve as local coordinates on the physical phase space. The corresponding symplectic structure is induced by (3.9)–(3.11). Another possibility would be to solve $F = 0$ with respect to, say, $x_N$ and $\varphi_2 = 0$ with respect to any of the momenta, say, $p_N$, i.e. $u_a = x_a, p_{u_a} = p_a,\; a = 1, 2, ..., N-1$.

Let us illustrate the procedure with the two-dimensional case, the motion on a plane constrained to a curve. We introduce the following parametrization of the constraint surface

$$\mathbf{x} = \mathbf{f}(u) \; , \quad \mathbf{p} = p_u \partial_u \mathbf{f} \; , \tag{3.12}$$

where $u$ is a parameter on the curve, note that $\boldsymbol{\partial} F \sim T \partial_u \mathbf{f}$ along the curve $F = 0$, where $T_{ij} = \varepsilon_{ij} = -\varepsilon_{ji},\; \varepsilon_{12} = 1$. The variables $u$ and $p_u$ are local coordinates on the physical phase space. We get

$$\Delta_{ab} = \{\varphi_a, \varphi_b\} = T_{ab} (\boldsymbol{\partial} F)^2 \; , \tag{3.13}$$

$$\Delta^{ab} = -T_{ab} (\boldsymbol{\partial} F)^{-2} \; . \tag{3.14}$$

Choosing some function $u = u(\mathbf{x})$ (e.g. one can simply invert the relation $x_1 = f_1(u)$) we obtain

$$p_u = \gamma(\mathbf{x}) (\mathbf{p}, T \boldsymbol{\partial} F) \; , \tag{3.15}$$

where $\gamma(\mathbf{x})$ depends on the choice of $u(\mathbf{x})$. Hence

$$\{u, p_u\}_D = \gamma(\mathbf{x}) (\boldsymbol{\partial} u(x), T \boldsymbol{\partial} F)|_{F=0} = \Gamma(u) \; . \tag{3.16}$$

The Darboux transformation for the physical symplectic form reads $(u, p_u) \to (u, p_u/\Gamma)$.

In general, the constraint surface may have a nontrivial topology, which must be taken into account when studying the dynamics in local symplectic coordinates. Consider, for

example, a particle on a circle, $F(\mathbf{x}) = \mathbf{x}^2 - R^2$. The radial motion is frozen and the other constraint is $(\mathbf{p}, \mathbf{x}) = 0$. A natural parametrization of the physical phase space (according to (3.12)) is

$$\mathbf{x} = Re^{\omega T}\boldsymbol{\chi} \ , \quad \mathbf{p} = p_\omega T\mathbf{x}(\omega)/R \tag{3.17}$$

where $\chi_i = \delta_{i1}$ and $\omega = \tan^{-1} x_2/x_1$, $p_\omega = (\mathbf{p}, T\mathbf{x})$. Calculating the Dirac bracket we obtain

$$\{\omega, p_\omega\}_D = 1 \ , \tag{3.18}$$

i.e. $\omega$ and $p_\omega$ are Darboux coordinates on the physical phase space which has the topology of a cylinder because $\omega$ is a cyclic (compact) variable, $\omega \in [-\pi, \pi)$. One can also choose $u = x_1$ and $p_u = p_1$ as local coordinates on the physical phase space . The corresponding symplectic structure assumes the form

$$\{u, p_u\}_D = 1 - \frac{u^2}{R^2} \ . \tag{3.19}$$

The Darboux transformation for (3.19) is

$$u = R\cos\omega \ , \quad p_u = -\frac{1}{R} p_\omega \sin\omega \ . \tag{3.20}$$

Now the topology of the physical phase space is hidden in singularities of the symplectic structure: It vanishes at $u = \pm R$.

The examples illustrates an arbitrariness in choosing a parametrization of the physical phase space: The parametrization is determined up to a general coordinate transformation on the physical phase space. The Dirac bracket ensures a covariance of the Hamiltonian dynamics with respect to such transformations. This covariance of the classical dynamics is lost upon canonical quantization as we now proceed to demonstrate.

## 4 Ambiguities in the canonical quantization of second-class constraints

The canonical quantization of a classical system implies that the canonical variables $\theta^i$ become hermitian operators $\hat{\theta}^i$ that act in a Hilbert space and obey the canonical commutation relations $[\hat{\theta}^i, \hat{\theta}^j] = i\hbar\{\hat{\theta}^i, \hat{\theta}^j\} = i\hbar \overset{\circ}{\omega}{}^{ij}$. The recipe is generally correct only in Cartesian coordinates [2]. Though the canonical variables $\theta^j$ have been assumed to be Cartesian, the quantization postulate should be modified when second-class constraints are present. The point is that the conditions $\hat{\varphi}_a = \varphi_a(\hat{\theta}) = 0$ can not be imposed on the operator level because they would be in conflict with the commutation relations $[\hat{\varphi}_a, \hat{\varphi}_b] \neq 0$: The constraints cannot be solved before calculating the commutation relations. The problem is resolved by replacing the Poisson bracket by the Dirac one in the canonical quantization postulate, that is [1]

$$[\hat{\theta}^i, \hat{\theta}^j] = i\hbar\{\theta^i, \theta^j\}_D|_{\theta=\hat{\theta}} = i\hbar\omega_D^{ij}(\hat{\theta}) \ . \tag{4.1}$$

Since the Dirac symplectic structure is degenerate, not every canonical variable can be made an operator (e.g. if $\varphi_1 = q = 0$, $\varphi_2 = p = 0$, then the Dirac rule leads to $[\hat{q}, \hat{p}] = 0$, i.e. the canonical variables are commuatative and, therefore, remain $c$-numbers upon quantization). For a generic second-class constrained system, the Dirac commutation relations (4.1) are constructed so that the operators of constraints commute with canonical variables $[\hat{\varphi}_a, \hat{\theta}^i] = 0$ and, hence, can be given any $c$-number value, in paticular, $\hat{\varphi}_a = 0$, enforcing the constraints on the quantum level. This comprises the geometrical meaning of the Dirac approach.

The recipe (4.1) is not however free of ambiguities either. The Dirac symplectic structure depends on the canonical variables and therefore the replacement $\theta^i$ by the corresponding operators usually leads to the operator ordering ambiguity.

An incorrect operator ordering in the right-hand side of (4.1) can break the associativity of the operator algebra (4.1) (the Jacobi identity is violated upon quantization). To restore the associativity, terms of higher orders of $\hbar$ should be added to the right-hand side of (4.1). Even after the associativity problem has been resolved in some way, one needs still to verify that this solution has not violated the quantization consistency conditions $[\hat{\theta}^i, \hat{\varphi}_a] = 0$. The latter would involve solving the operator ordering problem in the constraints in a way compatible with the operator ordering in the symplectic structure. In general this program may be very involved. After all the consistency problems have been resolved, one should face a not less difficult problem of constructing a representation of the algebra (4.1) in order to be able to calculate amplitudes (e.g. the evolution operator kernel).

As an alternative approach one can consider quantization after solving the constraints, meaning that the physical symplectic structure (2.5) is to be quantized. It should be noted that such an approach, though it resolves the operator ordering in the constraints, still has this problem in the symplectic structure (2.5). Going over to the Darboux variables does not help in this regard because canonical quantization is not generally correct in curvilinear coordinates as has been mentioned before. The problem appears even more serious if one notices that the Darboux coordinates are determined up to a canonical transformation, whereas canonical quantization and canonical transformations are not commutative operations. Thus, such a reduced phase-space quantization is coordinate dependent, and in this regard cannot be considered as a self-consistent quantization scheme.

As an illustration, consider the canonical quantization of the Dirac symplectic structure for a particle on a circle. Here $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$, and canonical quantization of the Dirac bracket (3.9) – (3.11) yields the following commutation relations

$$[\hat{x}_i, \hat{x}_j] = 0 \,, \tag{4.2}$$

$$[\hat{x}_j, \hat{p}_k] = i\hbar(\delta_{jk} - \frac{\hat{x}_j \hat{x}_k}{\hat{\mathbf{x}}^2}) \,, \tag{4.3}$$

$$[\hat{p}_j, \hat{p}_k] = i\hbar \frac{1}{\hat{\mathbf{x}}^2}(\hat{p}_j \hat{x}_k - \hat{p}_k \hat{x}_j) \,. \tag{4.4}$$

The operator ordering problem appears only in (4.4) and can be resolved by putting all the $\hat{p}_i$ either to the left or to the right of the $\hat{x}_i$ in the right-hand side of (4.4). Note that

$\hat{\mathbf{x}}^2$ commutes with all the canonical operators, so it does not matter where it is placed in the right-hand side of (4.4). One can check that this ordering is compatible with the hermiticity of the canonical operators and provides the Jacobi identity (associativity) in quantum theory. In this particular model, the operator ordering problem in the constraints is not of any relevance. Indeed, $\hat{\varphi}_1 = \hat{\mathbf{x}}^2 - R^2$ does not have any, and commutes with all the canonical operators. According to (4.3), the ordering correction to $\hat{\varphi}_2$ is a $c$-number and, hence, does not affect the relations $[\hat{\varphi}_2, \hat{x}_j] = [\hat{\varphi}_2, \hat{p}_j] = 0$.

The algebra (4.2)–(4.4) has a representation in a space of $2\pi$-periodic functions $\psi(\omega + 2\pi) = \psi(\omega)$

$$\begin{aligned}
\hat{x}_1\psi(\omega) &= R\cos\omega\psi(\omega)\ , \qquad \hat{x}_2\psi(\omega) = R\sin\omega\psi(\omega)\ ; && (4.5)\\
\hat{p}_1\psi(\omega) &= \frac{i\hbar}{2R}(\sin\omega\partial_\omega + \partial_\omega\sin\omega)\psi(\omega)\ ; && (4.6)\\
\hat{p}_2\psi(\omega) &= -\frac{i\hbar}{2R}(\cos\omega\partial_\omega + \partial_\omega\cos\omega)\psi(\omega)\ ; && (4.7)\\
\langle\psi_1|\psi_2\rangle &= \int_0^{2\pi} d\omega\, \psi_1^*\psi_2\ . && (4.8)
\end{aligned}$$

In this representation $\hat{\varphi}_1 = 0$ and $\hat{\varphi}_2$ is a $c$-number determined by the chosen operator ordering. The physical Hamiltonian assumes the form

$$\hat{H}_{ph} = \frac{1}{2R^2}\hat{p}_\omega^2 + \frac{\hbar^2}{8R^2}\ , \quad \hat{p}_\omega = -i\hbar\partial_\omega\ . \tag{4.9}$$

In addition to the kinetic energy opertor on the circle, the physical Hamiltonian contains a "quantum" potential that has occured through the Dirac degenerate commutation relations. For a generic manifold, the Dirac approach leads to a quantum potential that depends on position on the manifold.

A similar quantum potential was also predicted in the framework of the path integral qunatization on manifolds [3], and found to be proportional to the scalar curvature of the manifold. It is interesting to observe that canonical quantization of the Dirac bracket leads to a *different* prediction. For an $N$-dimensional sphere, the scalar curvature potential reads $\alpha\hbar^2 N(N-1)/R^2$ with $\alpha$ being a constant, i.e., it vanishes for a circle, whereas the embedding of the $N$-sphere into $\mathbb{R}^{N+1}$ and canonical quantization of the Dirac bracket (4.2) – (4.4) would yield the other form of the vacuum energy $\hbar^2N^2/8R^2$. Note that the algebra (4.2) – (4.4) applies to quantum motion on the $N$-sphere as has been shown above. Its representation is easy to find by going over to the spherical coordinates and thereby obtaining the physical Hamiltonian.

Let us turn to quantization in Darboux variables. The canonical quantization of the physical symplectic structure (2.5) in the Darboux coordinates would lead to a different Hamiltonian

$$\hat{H}'_{ph} = \frac{1}{2R^2}\hat{p}_\omega^2\ . \tag{4.10}$$

In this case the "extra" quantum potential is not unique at all. The source of troubles is that the Darboux variables (or any set of canonical coordinates parametrizing the

physical phase space) are determined only up to a general canonical transformation and, hence, are generally associated with non-Cartesian coordinates for which the canonical quantization is not generally consistent [2]. For instance, one can choose an alternative parametrization of the physical phase space by going over to new canonical coordinates $(\omega, p_\omega) \to (u = \sin\omega,\ p_u = p_\omega/\cos\omega)$, $\{u, p_u\} = 1$. After canonical quantization, $[\hat{u}, \hat{p}_u] = i\hbar$, the physical Hamiltonian would exhibit an operator ordering problem which has no unique solution. Given a particular operator ordering in the Hamiltonian, the change of variables $u = \sin\omega$ would not generally lead to (4.10), rather the quantum Hamiltonian will have an $\omega$ dependent potential of order $\hbar^2$. This quantum potential is fully determined by the physical phase space parametrization chosen. In this regard quantization of the Dirac degenerate symplectic structure (2.2) in a flat phase space looks more preferable because different parametrizations of the physical phase space are associated with different realizations of the same algebra of commutation relations for the canonical variables.

We would also like to mention that we do not consider the physical origin of the second-class constraints and assume the latter to be given a priori. In general, the applicability of the Dirac formalism to concrete physical sysytems can be questioned. For instance, the motion on a sphere can be physically interpreted as a motion in a thin spherical layer so that the radial motion is confined by a spherically symmetrical potential well. By going over to spherical coordinates in the free-particle Schrödinder equation, one can show that in the limit when the layer thickness is much less than the sphere radius, the quantum potential is given by the quantum centrifugal barrier $\hbar^2 N(N-2)/8R^2$ acting on a particle in the thin spherical layer, which is different from the one predicted by the Dirac or path integral approaches.

So, it is an open question whether or not the Dirac quantization scheme for second-class constraints can be applied to a particular dynamical system.

# 5 Quantization of second class constraints via the abelian conversion method

We have seen that the quantization of the Dirac bracket poses a few problems: the operator ordering in the commutation relation algebra (the associativity problem), the problem of finding a representation of the Dirac commutation relations and the ordering problem in the operators of constraints. Quantum dynamics depends on a particular solution to them and, generally, is not unique. The first two problems are most difficult. So one should develop a formalism that would allow one to avoid them. Such a formalism is known as the conversion of second-class constraints into first-class constraints by extending the original phase space by extra (gauge) degrees of freedom [4].

Let us extend the original phase space of a system with $2M$ independent second-class constraints by adding to it $2M$ independent variables $\phi^a$ with the canonical symplectic structure

$$\{\phi^a, \phi^b\} = \overset{\circ}{\omega}{}^{ab}, \quad \{\phi^a, \theta^i\} = 0 \ . \tag{5.1}$$

The original second-class constraints $\varphi_a(\theta)$ are then converted into abelian first-class con-

straints $\sigma_a = \sigma_a(\theta, \phi)$. Their explicit form is determined by a system of first-order differential equations

$$\{\sigma_a, \sigma_b\} = 0 \tag{5.2}$$

with the initial condition

$$\sigma_a(\theta, \phi = 0) = \varphi_a(\theta) \ . \tag{5.3}$$

A dynamical equivalence of the original second-class constrained system to the abelian gauge system is achieved by a specific extension of the original Hamiltonian of the system, $H_s(\theta) \to \bar{H}_s(\theta, \phi)$, such that

$$\{\bar{H}_s, \sigma_a\} = 0 \ , \quad \bar{H}_s(\theta, \phi = 0) = H_s(\theta) \ . \tag{5.4}$$

One can show that equations of motion generated by the extended action

$$\bar{S} = \int dt \left( \frac{1}{2}\phi^a \overset{\circ}{\omega}_{ab}\dot{\phi}_b + \frac{1}{2}\theta^i \overset{\circ}{\omega}_{ij}\dot{\theta}^j - \bar{H}_s - \bar{\lambda}^a\sigma_a \right) \ , \tag{5.5}$$

are equivalent to those generated by the original action

$$S = \int dt \left( \frac{1}{2}\theta^i \overset{\circ}{\omega}_{ij}\dot{\theta}^j - -H_s - \lambda^a\varphi_a \right) \ . \tag{5.6}$$

We remark that in contrast to the theory (5.6), the Lagrange multipliers $\bar{\lambda}^a$ in the gauge theory (5.5) are not determined by the equations of motion ( the matrix $\Delta_{ab} = \{\sigma_a, \sigma_b\}$ vanishes according to (5.2)). A choice of $\bar{\lambda}^a$ implies gauge fixing. In particular, one can always choose $\bar{\lambda}^a$ so that $\phi^a = 0$ on the constraint surface $\sigma_a = 0$ for all moments of time. With this choice the equations of motion of the system (5.5) become the equations of motion of the original system.

Equations (5.2) and (5.4) are not easy to solve for generic $\varphi_a$ and $H_s$. However, if at least one set of Darboux variables for the Dirac bracket is known, then the solution can be found explicitly [5]. In particular, for a particle on a circle we find

$$\sigma_1 = \varphi_1 + P = \mathbf{x}^2 - R^2 + P \ , \tag{5.7}$$

$$\sigma_2 = \varphi_2 + 2\mathbf{x}^2 Q = (\mathbf{x}, \mathbf{p}) + 2\mathbf{x}^2 Q \ , \tag{5.8}$$

where $\phi^1 = Q,\ \phi^2 = P$ and $\{Q, P\} = 1$. To find a solution to (5.4), one can make use of a simple observation that $\sigma_{1,2}$ and $p_\omega = (\mathbf{p}, T\mathbf{x})$ could be regarded as canonical momenta for $Q$, $\ln r/R$ and $\omega = \tan^{-1} x_2/x_1$, respectively. In these new canonical variables equation (5.4) is greatly simplified and we obtain

$$\bar{H}_s = \frac{1}{2}\left( \frac{\sigma_2^2}{\sigma_1 + R^2} + \frac{(\mathbf{p}, T\mathbf{x})^2}{\sigma_1 + R^2} \right) \ . \tag{5.9}$$

When $P = Q = 0$, $\sigma_1 + R^2 = \mathbf{x}^2$ and $\sigma_2 = (\mathbf{p}, \mathbf{x})$ so that $\bar{H}_s$ turns into the free-particle Hamiltonian $H_s = \mathbf{p}^2/2$ (recall that the vectors $\mathbf{x}$ and $T\mathbf{x}$ form an orthogonal basis on a

plane). We remark also that formulas (5.7)–(5.9) apply to an $N$-dimensional rotator (a motion on the $N$-dimensional sphere); one should only replace the squared total angular momentum of the rotator by its $N$-dimensional analog: $(\mathbf{p}, T\mathbf{x})^2 \to L^2 = \sum_a (\mathbf{p}, T_a\mathbf{x})^2$ where $T_a$ are $N \times N$ real antisymmetric matrices, generators of SO(N).

The first-class constraints (5.7) and (5.8) generate gauge transformations on the extended phase space which are translations of $Q$ and dilatation of the radial variables $r$, while the angular variables and its momenta remain invariant. Thus, $Q$ and $r$ are pure gauge degrees of freedom, and the angular variables comprise guage invariant physical degrees of freedom as expected.

The gauge transformations are also canonical transformations with generators being $\sigma_{1,2}$. An operator of finite canonical transformations generated by the constraints (5.7) and (5.8) has the form

$$\exp(\xi^a \mathrm{ad}\sigma_a) \ , \quad \mathrm{ad}\sigma_a = \{\sigma_a, \cdot\} \ . \tag{5.10}$$

Applying it to the canonical variables in the extended phase space, one finds

$$Q \ \to \ Q - \xi_1 = Q_\xi \ , \qquad P \to P + (1 - e^{-2\xi_2})\mathbf{x}^2 = P_\xi \ ; \tag{5.11}$$

$$\mathbf{x} \ \to \ e^{-\xi_2}\mathbf{x} = \mathbf{x}_\xi \ , \qquad \mathbf{p} \to e^{\xi_2}\mathbf{p} + 2\mathbf{x}(Qe^{\xi_2} - (Q - \xi_1)e^{-\xi_2}) = \mathbf{p}_\xi \ . \tag{5.12}$$

From (5.11) follows that there always exists a choice of the gauge parameters $\xi_a$ such that $Q = P = 0$ for all moments of time.

After the conversion has been made, the system can be quantized according to the Dirac method for the gauge theories. Namely, all the canonical variables of the extended Euclidean phase space become operators obeying the standard Heisenberg commutation relations

$$[\hat{\theta}^j, \hat{\theta}^k] = i\hbar \stackrel{\circ}{\omega}{}^{jk} \ , \ [\hat{\phi}^a, \hat{\phi}^b] = i\hbar \stackrel{\circ}{\omega}{}^{ab} \ , \tag{5.13}$$

while the operators of constraints $\hat{\sigma}_a$ select physical states

$$\hat{\sigma}_a \Psi_{ph} = 0 \ . \tag{5.14}$$

Equation (5.14) means that the physical states must be invariant under gauge transformations generated by first-class constraints. In particular, for the rotator we find that solutions to the Dirac constraint equations (5.14) are given by functions $\Psi_{ph} = f(Q, r)\psi(\omega)$ where $f(Q, r)$ is uniquely fixed by (5.14), while $\psi(\omega)$ is an arbitrary function of the polar angle on a plane. For the $N$-dimensional sphere, the physical Hilbert space consists of functions on the sphere. So, it is the Hilbert space of a quantum rotator as expected.

Thus, the problem of quantization of second class constraints has a natural gemetrical solution in the framework of the conversion method. The technical difficulties do not disappear completely; they are now associated with solving the conversion equations (5.2) and (5.4). Nevertheless, the approach may be simpler than the original Dirac approach. Even in the case of the rotator, the representation problem for the algebra (4.2)–(4.4) does not appear to be a feasible task if the geometrical origin of this algebra is unknown.

An important advantage of the conversion method is that it does not rely on any particular parametrization of the physical phase space. For this reason we shall adopt it

as starting point to develop a path integral formalism for second-class constrained systems which is invariant with respect to the parametrization choice of the physical phase space and, in this sense, to achieve coordinate independence of the quantum theory.

## 6 The projection method

We assume the operators $\hat{\sigma}_a$ to be hermitian and that they generate unitary transformations in the total Hilbert space. Since by construction they commute with the Hamiltonian, the total Hilbert space of an abelian gauge system obtained by the conversion procedure can always be split into an orthogonal sum of a subspace formed by gauge invariant states (5.14) and a subspace that consists of gauge variant states. Therefore an averaging of any state, being a linear combination of eigenstates of the Hamiltonian, over the abelian gauge group automatically leads to a projection operator onto the physical subspace of gauge invariant states:

$$\hat{\mathcal{P}} = \int \delta_\sigma \Omega e^{-i\Omega^a \hat{\sigma}_a} \tag{6.1}$$

where $\delta_\sigma \Omega$ is a normalized measure on the space of gauge transformation parameters. If the spectrum of the constraint operators $\hat{\sigma}_a$ is not discrete, then the parameters $\Omega^a$ range over a non-compact domain. In this case we adopt a certain regularization of the measure $\delta_\sigma \Omega$ which provides [6] (see also Section 9)

$$\int \delta_\sigma \Omega = 1 \tag{6.2}$$

and, hence, $\hat{\mathcal{P}}$ is the projection operator $\hat{\mathcal{P}}^2 = \hat{\mathcal{P}}$ such that

$$\hat{\mathcal{P}} \Psi_{ph} = \Psi_{ph} \ , \quad \hat{\mathcal{P}} \Psi_{nph} = 0 \tag{6.3}$$

for any gauge invariant state $\Phi_{ph}$ and any gauge variant state $\Psi_{nph}$ (by definition, $\hat{\sigma}_a \Psi_{nph} \neq 0$). Its kernel is determined as the gauge group average of the unit operator kernel

$$\langle \theta'', \phi'' | \theta', \phi' \rangle^{ph} \equiv \langle \theta'', \phi'' | \hat{\mathcal{P}} | \theta', \phi' \rangle = \int \delta_\sigma \Omega \langle \theta'', \phi'' | e^{-i\Omega^a \hat{\sigma}_a} | \theta', \phi' \rangle \ , \tag{6.4}$$

where $|\theta, \phi\rangle$ is the coherent state defined as

$$|\theta, \phi\rangle = \exp\left( i\theta^j \overset{\circ}{\omega}_{jk} \hat{\theta}^k + i\phi^a \overset{\circ}{\omega}_{ab} \hat{\phi}^b \right) |0\rangle \tag{6.5}$$

with $|0\rangle$ being the ground state of the harmonic oscillator.

Accordingly, the physical transition amplitude in the coherent-state representation is obtained from the unconstrained one by averaging the latter over the gauge group

$$\begin{aligned} \langle \theta'', \phi'', T | \theta', \phi' \rangle^{ph} &= \int \delta_\sigma \Omega \langle \theta'', \phi'', T | e^{-i\Omega^a \hat{\sigma}_a} | \theta', \phi' \rangle && (6.6) \\ &\equiv \int \frac{d\phi d\theta}{(2\pi)^{N+M}} \langle \theta'', \phi'', T | \theta, \phi \rangle \langle \theta, \phi | \hat{\mathcal{P}} | \theta', \phi' \rangle \ . && (6.7) \end{aligned}$$

The unconstrained transition amplitude is given by the coherent-state path integral

$$\langle\theta'',\phi'',T|\theta',\phi'\rangle = \int \mathcal{D}\theta\mathcal{D}\phi \exp\left(i\int\limits_0^T dt\left[\frac{1}{2}\theta^i\dot{\theta}_i + \frac{1}{2}\phi^a\dot{\phi}_a - \bar{h}_s(\theta,\phi)\right]\right) \tag{6.8}$$

with the boundary conditions $\theta(0) = \theta'$, $\theta(T) = \theta''$ and $\phi(0) = \phi'$, $\phi(T) = \phi''$; here $\theta_i = \overset{\circ}{\omega}_{ij}\theta^j$, $\phi_a = \overset{\circ}{\omega}_{ab}\phi^b$ and $\bar{h}_s$ is the lower symbol for the operator $\hat{\bar{H}}_s$

$$\hat{\bar{H}}_s = \int \frac{d\theta}{(2\pi)^N}\frac{d\phi}{(2\pi)^M}\bar{h}_s(\theta,\phi)|\theta,\phi\rangle\langle\phi,\theta| \ . \tag{6.9}$$

Dividing the time interval $T$ into $n$ pieces $\varepsilon = T/n$ and taking a convolution of $n$ kernels (6.7) where $T \to \varepsilon$, we arrive at the coherent-state path integral representation of the physical transition amplitude

$$\langle\theta'',\phi'',T|\theta',\phi'\rangle^{ph} = \int \mathcal{D}\theta\mathcal{D}\phi\mathcal{D}C(\omega)e^{i\int_0^T dt\left(\frac{1}{2}\theta^i\dot{\theta}_i + \frac{1}{2}\phi^a\dot{\phi}_a - \omega^a\sigma_a - \bar{h}_s\right)} \ . \tag{6.10}$$

The measure $\mathcal{D}C(\omega)$ for gauge variables, being the product of the local measures $\delta_\sigma\omega^a(t)$, provides the projection at each moment of time. The action in the exponential in (6.10) coincides with the classical action (5.6) up to possible operator ordering terms $\bar{H}_s - \bar{h}_s = O(\hbar)$. It is invariant with respect to gauge transformations generated by $\sigma_a$

$$\delta\theta^i = \xi^a \mathrm{ad}\sigma_a\theta^i \ , \qquad \delta\phi^a = \xi^b \mathrm{ad}\sigma_b\phi_a \ , \qquad \delta\omega^a = -\dot{\xi}^a \ , \tag{6.11}$$

where the infinitesimal functions of time $\xi^a$ satisfy zero boundary conditions

$$\xi^a(0) = \xi^a(T) = 0 \ , \tag{6.12}$$

which ensure that the boundary terms occurring upon varying $\bar{S}$ vanish.

To obtain the corresponding path integral on the physical phase space, one usually has to integrate out all the gauge variables $\omega$ and non-physical degrees of freedom. Formally, it can be achieved by going over to new canonical variables such that the abelian constraints $\sigma_a$ become new canonical momenta. Due to the gauge invariance the Hamiltonian $\bar{h}_s$ is independent of the corresponding canonical coordinates. Since the Liouville measure constituting the formal path integral measure is invariant under canonical transformations, the integration over non-physical variables becomes trivial. Yet, the gauge transformations in the new variables are translations of canonical coordinates for the new momenta $\sigma_a$, so the gauge average can also be done explicitly. Having done this, one seems to obtain a path integral over the physical phase space parametrized by a certain set of canonical variables. Note that the canonical transformation discussed above is not unique and is determined only up to a general canonical transformation on the physical phase space. On the other hand, we have seen in section 4 that quantum theory may well depend on a particular parametrization of the physical phase space, which is in conflict with the formal coordinate invariance of the path integral measure.

To explain the contradiction we observe that the above procedure of reducing the path integral measure onto the physical phase space relies on the formal invariance of the conventional Liouville measure with respect to canonical transformation. Unfortunately, this is a wrong assumption. A typical example is a Hamiltonian dynamics on a phase-space plane. By a canonical transformation one can always locally turn the Hamiltonian into a free particle Hamiltonian. So assuming a formal invariance of the path integral measure with respect to general canonical transformations we would arrive to a contradiction that every quantum dynamical system with one degree of freedom is equivalent to a free particle.

Thus, in order to obtain a path integral on the physical phase space, the path integral measure in the amplitude (6.10) should be regularized in a way that provides a true covariance of the path integral (6.10) with respect to general canonical transformation.

# 7 The Wiener measure regularized path integral

For Hamiltonian systems without constraints a regularization of the path integral measure can be achieved by replacing the conventional Liouville measure by a pinned Wiener measure on continuous phase-space paths. The Wiener measure regularized phase space path integral for a general phase function $G(p,q)$ is then given by [7]

$$\begin{aligned}
&\lim_{\nu\to\infty} \mathcal{M}_\nu \int \exp\{i{\textstyle\int_0^T}[p_j\dot{q}^j + \dot{G}(p,q) - h(p,q)]\,dt\} \\
&\qquad\qquad \times \exp\{-(1/2\nu){\textstyle\int_0^T}[\dot{p}^2+\dot{q}^2]\,dt\}\,\mathcal{D}p\,\mathcal{D}q \\
&\quad = \lim_{\nu\to\infty}(2\pi)^N e^{N\nu T/2}\int \exp\{i{\textstyle\int_0^T}[p_j dq^j + dG(p,q) - h(p,q)dt]\}\,d\mu_W^\nu(p,q) \\
&\quad = \langle p'', q''|e^{-i\mathcal{H}T}|p',q'\rangle \quad ,
\end{aligned} \tag{7.1}$$

where the last relation involves a coherent state matrix element. Here we use the convention adopted in Section 1 that $q^j = \theta^{2j-1}$ and $p_j = \theta^{2j}$ (cf. (6.5)). In this expression we note that $\int p_j\,dq^j$ is a *stochastic integral*, and as such we need to give it a definition. As it stands both the Itô (nonanticipating) rule and the Stratonovich (midpoint) rule of definition for stochastic integrals yield the same result (since $dp_j(t)dq^k(t) = 0$ is a valid Itô rule in these coordinates). Under any change of canonical coordinates, we consistently will interpret this stochastic integral in the Stratonovich sense because it will then obey the ordinary rules of calculus. We also emphasize the covariance of this expression under canonical coordinate transformations. In particular, if $\overline{p}d\overline{q} = pdq + dF(\overline{q},q)$ characterizes a canonical transformation from the variables $p,q$ to $\overline{p},\overline{q}$, then with the Stratonovich rule the path integral becomes

$$\begin{aligned}
&\langle \overline{p}'', \overline{q}''|e^{-i\mathcal{H}T}|\overline{p}',\overline{q}'\rangle \\
&\quad = \lim_{\nu\to\infty}(2\pi)^N e^{N\nu T/2}\int \exp\{i{\textstyle\int_0^T}[\overline{p}_j d\overline{q}^j + d\overline{G}(\overline{p},\overline{q}) - \overline{h}(\overline{p},\overline{q})dt]\}\,d\mu_W^\nu(\overline{p},\overline{q}) \\
&\quad = \lim_{\nu\to\infty}\mathcal{M}_\nu \int \exp\{i{\textstyle\int_0^T}[\overline{p}_j\dot{\overline{q}}^j + \dot{\overline{G}}(\overline{p},\overline{q}) - \overline{h}(\overline{p},\overline{q})dt]\} \\
&\qquad\qquad\qquad \times \exp\{-(1/2\nu){\textstyle\int_0^T}[d\sigma(\overline{p},\overline{q})^2/dt^2]\,dt\}\,\mathcal{D}\overline{p}\,\mathcal{D}\overline{q}\,,
\end{aligned} \tag{7.2}$$

where $\overline{G}$ incorporates both $F$ and $G$. In this expression we have set $d\sigma(\overline{p},\overline{q})^2 = dp^2 + dq^2$, namely, the new form of the flat metric in curvilinear phase space coordinates. We emphasize that this path integral regularization involves Brownian motion on a flat space whatever choice of coordinates is made. Our transformation has also made use of the formal – and in this case valid – invariance of the Liouville measure.

In order to fulfill our program of a coordinate-free path integral representation of second-class constrained systems, we have to extend the Wiener measure regularization to such systems.

## 8 The Wiener measure for second-class constraints

The regularized measure in the path integral (6.10) is obtained by the replacement

$$\mathcal{D}\theta\mathcal{D}\phi\mathcal{D}C(\omega) \to \mathcal{D}C(\omega)d\mu_W^g(\theta,\phi,\omega)\ , \tag{8.1}$$

where the gauged Wiener measure $d\mu_W^g$ is to be found. The Wiener measure regularization of the path integral should not violate gauge invariance, therefore, we impose the condition

$$\delta d\mu_W^g(\theta,\phi,\omega) = 0\ , \tag{8.2}$$

where the operator $\delta$ is determined in (6.11). Since the Wiener measure provides for covariance of the path integral (6.10) relative to canonical transformations, we perform a canonical transformation in (6.10) such that $\sigma_a$ become new canonical momenta

$$(\theta^i, \phi^a) \to (\pi_a = \sigma_a, y^a, \vartheta^\alpha)\ ; \tag{8.3}$$

here $y^a$ are canonical coordinates for $\pi_a$ and $\vartheta^a$ are canonical symplectic variables on the physical phase space (Darboux variables for the physical symplectic structure (2.5)). In the new variables the action assumes the form

$$\bar{S} = \int_0^T \left(\pi_a dy^a + \tfrac{1}{2}\vartheta^\alpha d\vartheta_\alpha - \omega^a\pi_a dt - dG - \bar{h}_s dt\right) \tag{8.4}$$

and for the Wiener measure we get

$$d\mu_W^g(\theta,\phi,\omega) = d\bar{\mu}_W^g(\pi,y,\vartheta,\omega)\ . \tag{8.5}$$

The gauge transformations (6.11) leave all the new variables untouched except the $y^a$ which are shifted

$$\delta\pi_a = 0\ ,\qquad \delta\vartheta^\alpha = 0\ ,\qquad \delta y^a = -\xi^a\ . \tag{8.6}$$

By the change of integration variables

$$y^a(t) \to y^a(t) - \int_t^T dt'\omega^a(t')\ , \tag{8.7}$$

one can remove the dependence on $\omega^a$ of the integral (6.10) for all intermediate moments of time $0 < t < T$. However the initial values of $y^a$ are not integration variables and

therefore the average over $\omega^a$ does not disappear without a trace. Let us make a change of gauge variables $\omega^a \to \dot{\omega}^a$, where the new variables $\omega^a$ satisfy the boundary condition $\omega^a(T) = 0$. Note that we are free to add any constant to $\omega^a$ because it does not affect the derivative $\dot{\omega}^a$; the boundary condition fixes this arbitrariness, providing a one-to-one correspondence between the old and new gauge integration variables. With this choice equation (8.7) assumes a simple form $y^a(t) \to y^a(t) + \omega^a(t)$. At the boundary $t = 0$ we have

$$y^a(0) \to y^a(0) + \Omega^a \ , \quad \omega^a(0) = \Omega^a \ . \tag{8.8}$$

Therefore the average measure for gauge variables is reduced to a single average over $\Omega^a$,

$$\mathcal{D}C(\dot{\omega}) \to \delta_\sigma \Omega \tag{8.9}$$

because $\int \mathcal{D}C(\dot{\omega}) = 1$.

Equation (8.2) is easy to solve in the new canonical variables

$$d\bar{\mu}_W^g(\pi, y, \vartheta, \omega) = d\tilde{\mu}_W^g(\pi, y - \omega, \vartheta) \ , \tag{8.10}$$

where the gauge variables $\omega$ have been replaced by their time derivatives as in (8.8). Note that under gauge transformation $\delta(y^a - \omega^a) = - - \xi^a - \delta\omega^a = 0$ because $\delta\omega^a = -\xi^a$ in accordance with (6.11) and the replacement $\omega^a \to \dot{\omega}^a$. Clearly, the further transformation of the $y$-integral to the new variables $y - \omega$ removes the dependence of the Wiener measure on the gauge variables for all intermediate moments of time, i.e.,

$$d\tilde{\mu}_W^g(\pi, y - \omega, \vartheta) \to d\tilde{\mu}_W^g(\pi, y, \vartheta) \tag{8.11}$$

in the path integral (6.10). As a result of these two canonical transformations the entire dependence of the path integral measure on gauge variables is reduced to a single average over a gauge orbit of the initial phase-space point with some *phase factor* determined by the phase function $\int dG$ of the canonical transformation (8.3). That is, we have recovered the projection formula (6.7) where the projection operator kernel is given by

$$\langle \theta, \phi | \theta', \phi' \rangle^{ph} = \int \delta_\sigma \Omega \langle \theta, \phi | \theta'_\Omega, \phi'_\Omega \rangle e^{i\bar{G}(\theta', \phi', \Omega)} \ , \tag{8.12}$$

with $\theta_\Omega$ and $\phi_\Omega$ being gauge transformations of the extended phase space variables generated by (5.10) with $\xi = \Omega$ and $\bar{G}$ is $G(t = 0)$ written in the initial canonical variables.

The Wiener measure regularized path integral (6.8) involved in the projection formula (6.7) should have the flat Wiener measure on the extended phase space according to our consideration in the previous section, that is,

$$d\tilde{\mu}_W^g(\pi, y, \vartheta) = d\mu_W^\nu(\theta, \phi) \ . \tag{8.13}$$

Having established the relation between $d\tilde{\mu}_W^g$ and the flat-space Wiener measure $d\mu_W^\nu$, we can perform a canonical transformation inverse to (8.3) to restore the dependence of the

Wiener measure on the gauge variables and thereby to find an explicit form of the desired gauged Wiener measure (8.1). Combining (8.5), (8.10) and (8.13) we conclude that

$$d\mu_W^g(\theta,\phi,\omega) = d\mu_W^\nu(\theta_\omega,\phi_\omega)\ , \tag{8.14}$$

where

$$\theta_\omega^i = e^{\omega^a \mathrm{ad}\sigma_a}\theta^i\ , \qquad \phi_\omega^a = e^{\omega^b \mathrm{ad}\sigma_b}\phi^a \tag{8.15}$$

with $\omega^a = \omega^a(t)$. The gauge invariance of the gauged Wiener measure (8.14) follows from the simple observation that

$$\delta\theta_\omega^i = \delta\phi_\omega^a = 0 \tag{8.16}$$

under the gauge transformation (6.11) where $\delta\omega \equiv --\xi$ according to the change of gauge variables $\omega \to \dot\omega$.

Thus, the Wiener measure regularized path integral for second-class constrained theories has the form

$$\begin{aligned} \langle\theta'',\phi'',T|\theta',\phi'\rangle^{ph} &= \int \mathcal{D}C(\omega)\int d\mu_W^\nu(\theta_\omega,\phi_\omega) e^{i\int_0^T dt\left(\frac{1}{2}\theta^i\dot\theta_i + \frac{1}{2}\phi^a\dot\phi_a - \dot\omega^a\sigma_a - \bar h_s\right)}\ ; & (8.17)\\ d\mu_W^\nu(\theta_\omega,\phi_\omega) &= e^{(N+M)\nu T/2}\mathcal{D}\theta\mathcal{D}\phi\exp\left(--\frac{1}{2\nu}\int_0^T dt(\dot\theta_\omega^2 + \dot\phi_\omega^2)\right)\ , & (8.18)\end{aligned}$$

where the limit $\nu\to\infty$ must be taken after calculating the path integral (8.17). In general, the gauged Wiener measure (8.18) depends not only on $\dot\omega$ but also on $\omega$ themselves, therefore, it is not possible to remove the dependence of the action in (8.17) on the time derivatives of the gauge variables by changing the gauge variables back $\dot\omega\to\omega$ in the average measure $\mathcal{D}C(\omega)$, while maintaining the locality of the gauged Wiener measure (8.18).

As an example consider the gauged Wiener measure for the two-dimensional rotator (a generalization to the $N$-dimensional case is trivial as remarked after Eq. (5.9)). The canonical transformation (8.3) can be chosen as

$$\begin{aligned} \pi_1 &= \sigma_1\ , \qquad y^1 = Q\ ; & (8.19)\\ \pi_2 &= \sigma_2\ , \qquad y^2 = \ln(|\mathbf{x}|/R)\ ; & (8.20)\\ \vartheta^2 &= (\mathbf{p}, T\mathbf{x})\ , \quad \vartheta^1 = \tan^{-1}(x_2/x_1)\ , \quad \{\vartheta^1,\vartheta^2\} = 1\ . & (8.21)\end{aligned}$$

As expected from $\mathrm{ad}\sigma_a \bar H_s = 0$, the canonical coordinates $y^a$ are cyclic. The gauge transformations

$$y^1 \to y^1 - \xi^1\ , \qquad y^2 \to y^2 - \xi^2 \tag{8.22}$$

induce gauge transformations of the initial canonical variables (5.11). Setting in (5.11) $\xi = \omega$ we obtain the gauged flat metric on the extended phase space that determines the Wiener measure

$$\begin{aligned} \int_0^T dt(\dot\theta_\omega^2 + \dot\phi_\omega^2) &= \int_0^T dt\left(\dot{\mathbf{p}}_\omega^2 + \dot{\mathbf{x}}_\omega^2 + \dot P_\omega^2 + \dot Q_\omega^2\right) & (8.23)\\ &= \int_0^T dt g_{AB}(\Lambda)\dot\Lambda^A\dot\Lambda^B\ , & (8.24)\end{aligned}$$

where $\Lambda^A$ denotes the set of all canonical and gauge variables $(\theta, \phi, \omega) = (\mathbf{p}, \mathbf{x}, P, Q, \omega)$; for the $N$-dimensional rotator, $\mathbf{p}$ and $\mathbf{x}$ are $N$-dimensional vectors in (5.11) and (8.24). Note that the metric $g_{AB}$ depends generally on all the $\Lambda^A$, as well as the components $g_{A\omega}$ and $g_{\omega\omega}$ do not vanish. Thus, the Wiener measure depends on gauge variables and their time derivatives.

Expression (8.24) holds for general second-class constrained systems. Its geometrical meaning is transparent. The metric $g_{AB}$ is, by construction, degenerate along the directions traversed by gauge transformations of the $\Lambda$. Hence the gauged Wiener measure describes a Brownian motion (with diffusion constant that tends to infinity) in the directions transverse to the gauge orbits, while the average over the gauge variables with the measure $\mathcal{D}C(\omega)$ regularizes the path integral along the gauge orbits. An explicit construction of this measure is discussed in section 9.

An unusual feature of the integral (8.17) is the appearance of the time derivatives of the gauge variables in the classical action. This was the price we paid for locality of the Wiener measure. One should realize that this is not always the case for the Wiener measure in gauge theories. If the canonical transformations generated by first-class constraints were linear and preserving a bilinear positive form on the extended phase space, then the associated Wiener measure (8.18) would have had no dependence on $\omega$ but on $\dot{\omega}$ only. The latter occurs for Yang-Mills type gauge theories [8]. In this case the dependence on $\dot{\omega}$ can be removed by a simple change of variables $\dot{\omega} \to \omega$ in the gauge average integral without violating the locality of the Wiener measure.

At first sight, the presence of the time derivatives of the gauge variables in the classical action seems to allow for non-physical motion with $\sigma = const \neq 0$ (a variation of the action relative to $\omega$ leads to the equation of motion $\dot{\sigma} = 0$ rather than just $\sigma = 0$). One has however to bear in mind that equation (8.17) describes a quantum motion whose gauge invariance is ensured by an appropriate average over the gauge variables. As long as the measure $\mathcal{D}C(\omega)$ provides at least one gauge group average in the time interval $0 \leq t \leq T$, contributions of states with $\sigma = const \neq 0$ are projected out from the transition amplitude in full accordance with the projection formula (6.7).

# 9 The average measure for gauge variables

The spectrum of the first-class constraint operators that usually occur upon the abelian conversion of second-class constraint operators is continuous. Therefore the average measure $\delta_\sigma \Omega$ in the projection operator (6.1) should be regularized to provide the normalization condition (6.2). Since the converted constraints are abelian, the projection operator (6.1) is the product of the projection operators for each independent abelian generator $\hat{\sigma}_a$. The latter allows us to treat the measure $\delta_\sigma \Omega$ as the product of normalized measures for each independent gauge variables $\Omega^a$. So we can drop the index $a$ and consider the measure only for one generator $\hat{\sigma}$.

The gauge transformations are translations of the gauge parameter $\Omega$. Hence any regularization (a cut-off) of the translation invariant measure $d\Omega$ would break the translation invariance and therefore an explicit gauge invariance of the path integral (8.17). In this

sense, the regularization would lead to a "gauge-fixing" term in the effective action in the integrand in (8.17). The gauge invariance of the amplitude (8.17) is guaranteed as long as the regularized measure for gauge variables provides at least one projection onto the physical subspace in the time interval $t \in [0, T]$.

Consider the regularized measure of the following form

$$\delta_\sigma \Omega = \sqrt{\frac{m}{2\pi}}\; e^{-\frac{m}{2}\Omega^2} d\Omega \ , \qquad m \to 0 \ . \tag{9.1}$$

Here $m$ is the regularization parameter. Clearly, the measure (9.1) is normalized to unity. Let $|\sigma\rangle$ be an eigenvector of the generator $\hat{\sigma}$. Applying the projector (6.1) to it we find

$$\begin{aligned} \hat{\mathcal{P}}|\sigma\rangle &= \int \delta_\sigma \Omega e^{-i\Omega\hat{\sigma}} |\sigma\rangle \\ &= \sqrt{\frac{m}{2\pi}} \int\limits_{-\infty}^{\infty} d\Omega\; e^{-\frac{m}{2}\Omega^2 - i\Omega\sigma} |\sigma\rangle \\ &= e^{-\frac{\sigma^2}{2m}} |\sigma\rangle \ . \end{aligned} \tag{9.2}$$

Taking the limit $m \to 0$ in (9.2) we see that for a hermitian operator $\hat{\sigma}$, the operator $\hat{\mathcal{P}}$ annihilates all eigenvectors of the gauge generator $\hat{\sigma}$ unless $\sigma = 0$. In the latter case $\hat{\mathcal{P}}$ acts as the unit operator, that is, it is the projector on the physical subspace.

Adopting the above regularization of the gauge average measure, we replace $T$ in the projection formula (6.6) by an infinitesimal time interval $\varepsilon = T/n$ and construct a convolution on $n$ infinitesimal propagators (6.6). The result has the form (6.10) where the gauge variable measure is

$$\mathcal{D}C(\omega) = \prod_{j=0}^{n-1} \sqrt{\frac{m}{2\pi}} e^{-\frac{m}{2}\omega_j^2} d\omega_j = \mathcal{N} e^{-\int_0^T dt (m\omega^2/2)} \prod_t \sqrt{m}\; d\omega(t) \ , \tag{9.3}$$

where $\omega(0) = \Omega$ (to match the notations in (6.6)). To take the continuum limit we have rescaled the gauge variables $\omega_j \to \sqrt{\varepsilon}\omega_j$ with $\varepsilon$ being the time slicing so that $\omega_j = \omega(t_j)$ and $\omega(t_{j+1}) = \omega(t_j + \varepsilon)$.

To make the gauged Wiener measure a local functional of gauge variables, it was proposed in section 8 to change the integration variables $\omega(t) \to \dot{\omega}(t)$. In the time-slice approximation of the path integral the change of gauge variables assumes the form

$$\omega_j \to (\omega_{j+1} - \omega_j)/\varepsilon \ , \qquad j = 0, 1, ..., n-1 \ . \tag{9.4}$$

The "extra" new variable $\omega_n = \omega(T)$ is fixed by the boundary condition $\omega(T) = 0$ as suggested in Section 8. In the new variables the measure for gauge variables turns into a flat-space Wiener measure for continuous paths pinned at one point

$$\mathcal{D}C_m(\dot{\omega}) = \mathcal{N} \exp\left(-\frac{m}{2}\int_0^T dt\; \dot{\omega}^2\right) \prod_t \sqrt{m}\; d\omega(t) \ , \qquad \omega(T) = 0 \ , \tag{9.5}$$

where the index $m$ stands to emphasize the dependence of the measure on the regularization parameter $m$. With this choice of the measure for gauge variables, we arrive at our coordinate-free and mathematically well-defined formulation for the path integral representation of the second class constrained systems.

To conclude the discussion, we note that by construction the limits $m \to 0$ and $\nu \to \infty$ commute in the integral (8.17). The latter follows from the projection formula (8.12) to which the path integral can be transformed by a change of variables as has been shown in Section 8. The amlitude $\langle \theta, \phi | \theta'_\Omega, \phi'_\Omega \rangle$ can be calculated at a finite $\nu$. For any fixed $\Omega$ it is a regular function of $\nu$ in the vicinity of $\nu = \infty$. So, taking its gauge invariant part either before the limit $\nu \to \infty$ or after it would yield the same result. It is convenient then to make a particular choice of the parameter $m$ to simplify the path integral measure form. Namely, we set

$$m = 1/\nu \ , \tag{9.6}$$

so that the path integral measure would depend only on one parameter to be taken to infinity after performing the sum over paths. Thus, the gauged Wiener measure assumes the following (unified) form

$$d\mu_W^\nu(\theta_\omega, \phi_\omega)\mathcal{D}C_{1/\nu}(\dot{\omega}) = e^{(N+M)\nu T/2} e^{-\frac{1}{2\nu}\int_0^T dt\left(\dot{\theta}_\omega^2 + \dot{\phi}_\omega^2 + \dot{\omega}^2\right)} \mathcal{D}\theta \mathcal{D}\phi \mathcal{D}\omega \ . \tag{9.7}$$

As has been shown in Section 8, by a suitable change of variables one can always remove the dependence of the integrand in (8.17) on the gauge variables $\omega^a$ for all moments of time except $t = 0$. Then the integral over $\omega^a$ yields the kernel of the imaginary time transition amplitude for a free motion in the M-dimensional Euclidean space of gauge parameters where $\omega^a(T) = 0$ and $\omega^a(0) = \Omega^a$. The integration over the initial values $\Omega^a$ weighted with this kernel is precisely the gauge average (6.6) regularized as prescribed by (9.2). The measure (9.7) describes a Brownian motion with an arbitrarily large diffusion constant on the unified space $(\theta, \phi, \omega)$, and, in this sense, all degrees of freedom, physical and gauge ones, are treated on equal footing in the Wiener measure (9.7).

We note that Ashworth has also studied Wiener measure regularizations for systems with first class constraints [9].

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